Exercise 9.2.1

Find the general solutions of the PDEs in Exercises 9.2.1 to 9.2.4.

$$\frac{\partial \psi}{\partial x} + 2\frac{\partial \psi}{\partial y} + (2x - y)\psi = 0.$$

Solution

Since ψ is a function of two variables $\psi = \psi(x, y)$, its differential is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy.$$

Dividing both sides by dx, we obtain the relationship between the total derivative of ψ and the partial derivatives of ψ.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{dy}{dx}\frac{\partial \psi}{\partial y}$$

In light of this, the PDE reduces to the ODE,

$$\frac{d\psi}{dx} + (2x - y)\psi = 0, \qquad (1)$$

along the characteristic curves in the xy-plane that satisfy

$$\frac{dy}{dx} = 2, \qquad y(0, \xi) = \xi, \qquad (2)$$

where ξ is a characteristic coordinate. Integrate both sides of equation (2) with respect to x to solve for $y(x, \xi)$.

$$y(x, \xi) = 2x + \xi$$

From this equation we see that $2x - y = -\xi$, which means equation (1) becomes

$$\frac{d\psi}{dx} - \xi\psi = 0.$$

Solve this ODE by separating variables.

$$\frac{d\psi}{\psi} = \xi\, dx$$

Integrate both sides.

$$\int \frac{d\psi}{\psi} = \int \xi\, dx$$

$$\ln|\psi| = \xi x + f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate ξ. Exponentiate both sides.

$$\begin{aligned} |\psi| &= e^{\xi x + f(\xi)} \\ &= e^{f(\xi)} e^{\xi x} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\psi(x, \xi) = \pm e^{f(\xi)} e^{\xi x}$$

Use a new arbitrary function $g(\xi)$ for $\pm e^{f(\xi)}$.

$$\psi(x, \xi) = g(\xi) e^{\xi x}$$

Therefore, since $\xi = y - 2x$,

$$\psi(x, y) = g(y - 2x) e^{x(y - 2x)}.$$